DENJOY-CARLEMAN DIFFERENTIABLE PERTURBATION OF POLYNOMIALS AND UNBOUNDED OPERATORS

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Abstract. Let $t \mapsto A(t)$ for $t \in T$ be a C^M-mapping with values unbounded operators with compact resolvents and common domain of definition which are self-adjoint or normal. Here C^M stands for C^ω (real analytic), a quasianalytic or non-quasianalytic Denjoy-Carleman class, C^∞, or a Hölder continuity class $C^{0,\alpha}$. The parameter domain T is either $\mathbb{R}$ or $\mathbb{R}^n$ or an infinite dimensional convenient vector space. We prove and review results on C^M-dependence on t of the eigenvalues and eigenvectors of $A(t)$.

Theorem. *Let $t \mapsto A(t)$ for $t \in T$ be a parameterized family of unbounded operators in a Hilbert space H with common domain of definition and with compact resolvent.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint then the following holds:

(A) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.*
(B) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized C^Q in t.*
(C) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^L in t.*
(D) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^∞ in t.*
(E) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in t.*
(F) *If $A(t)$ is $C^{1,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ may be parameterized in a C^1 way in t.*

If $t \in T = \mathbb{R}$ and all $A(t)$ are normal then the following holds:

(G) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized real analytically in s near $s = 0$.*
(H) *If $A(t)$ is C^Q in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^Q in s near $s = 0$.*
(I) *If $A(t)$ is C^L in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two of the unequal continuously arranged eigenvalues (see* [10, II.5.2]*) meet of infinite order, there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^L in s near $s = 0$.*

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(J) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue λ of $A(t_0)$ at which no two of the unequal continuously arranged eigenvalues (see* [10, II.5.2]*) meet of infinite order, there exists $N \in \mathbb{N}$ such that the eigenvalues near λ of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^∞ in s near $s = 0$.*

(K) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and no two of the unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized by absolutely continuous functions, locally in t.*

If $t \in T = \mathbb{R}^n$ and all $A(t)$ are normal then the following holds:

(L) *If $A(t)$ is C^ω or C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue λ of $A(t_0)$, there exist a finite covering $\{\pi_k : U_k \to W\}$ of a neighborhood W of t_0, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s. If A is self-adjoint, then we do not need power substitutions.*

(M) *If $A(t)$ is C^ω or C^Q in $t \in \mathbb{R}^n$, then the eigenvalues and their eigenvectors of $A(t)$ can be parameterized by functions which are special functions of bounded variation (SBV), see* [7] *or* [2]*, locally in t.*

If $t \in T \subseteq E$, a c^∞-open subset in an infinite dimensional convenient vector space then the following holds:

(N) *For $0 < \alpha \le 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are self-adjoint, then the eigenvalues of $A(t)$ may be parameterized in a $C^{0,\alpha}$ way in t.*

(O) *For $0 < \alpha \le 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are normal, then we have: For each $t_0 \in T$ and each eigenvalue z_0 of $A(t_0)$ consider a simple closed C^1-curve γ in the resolvent set of $A(t_0)$ enclosing only z_0 among all eigenvalues of $A(t_0)$. Then for t near t_0 in the c^∞-topology on T, no eigenvalue of $A(t)$ lies on γ. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$ be the N-tuple of all eigenvalues (repeated according to their multiplicity) of $A(t)$ inside of γ. Then $t \mapsto \lambda(t)$ is $C^{0,\alpha}$ for t near t_0 with respect to the non-separating metric*

$$d(\lambda, \mu) = \min_{\sigma \in \mathcal{S}_N} \max_{1 \le i \le N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of N-tuples.

Part (A) is due to Rellich [20] in 1942, see also [3] and [10, VII, 3.9]. Part (D) has been proved in [1, 7.8], see also [11, 50.16], in 1997, which contains also a different proof of (A). (E) and (F) have been proved in [12] in 2003. (G) was proved in [18, 7.1]; it can be proved as (H) with some obvious changes, but it is not a special case since C^ω does not correspond to a sequence which is an $\mathcal{L}$-intersection (see [14]). (J) and (K) were proved in [18, 7.1]. (N) was proved in [15].

The purpose of this paper is to prove the remaining parts (B), (C), (H), (I), (L), (M), and (O).

Definitions and remarks. Let $M = (M_k)_{k \in \mathbb{N} = \mathbb{N}_{\ge 0}}$ be an increasing sequence ($M_{k+1} \ge M_k$) of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for each compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \le C\, \rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

Here $Q = (Q_k)_{k\in\mathbb{N}}$ is a sequence as above which is quasianalytic, log-convex, and which is also an $\mathcal{L}$-intersection, see [14] or [13] and references therein. Moreover, $L = (L_k)_{k\in\mathbb{N}}$ is a sequence as above which is non-quasianalytic and log-convex.

That $A(t)$ is a real analytic, C^M (where M is either Q or L), C^∞, or $C^{k,\alpha}$ family of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ in the self-adjoint case, or $A(t)$ has closed graph and $A(t)A(t)^* = A(t)^*A(t)$ wherever defined in the normal case. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is of the respective differentiability class for each $u \in V$ and $v \in H$. ¿From now on we treat only $C^M = C^\omega$, C^M for $M = Q$, $M = L$, and $C^M = C^{0,\alpha}$.

This implies that $t \mapsto A(t)u$ is of the same class $C^M(E, H)$ (where E is either $\mathbb{R}$ or $\mathbb{R}^n$) or is in $C^{0,\alpha}(E, H)$ (if E is a convenient vector space) for each $u \in V$ by [11, 2.14.4, 10.3] for C^ω, by [13, 3.1, 3.3, 3.5] for $M = L$, by [14, 1.10, 2.1, 2.3] for $M = Q$, and by [11, 2.3], [9, 2.6.2] or [8, 4.14.4] for $C^{0,\alpha}$ because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

A sequence of functions λ_i is said to *parameterize the eigenvalues, if for each $z \in \mathbb{C}$ the cardinality $|\{i : \lambda_i(t) = z\}|$ equals the multiplicity of z as eigenvalue of $A(t)$.*

Let X be a C^ω or C^Q manifold. A *local blow-up* Φ over an open subset U of X means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \to U$ with center a C^ω or C^Q submanifold and of the inclusion $\iota : U \to X$. A *local power substitution* is a mapping $\Psi : V \to X$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \to X$ is the inclusion of a coordinate chart W of X and $\psi : V \to W$ is given by

$$(y_1, \dots, y_q) = ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0, 1\}^q$, where $y_1, \dots, y_q$ denote the coordinates of W (and $q = \dim X$).

This paper became possible only after some of the results of [13] and [14] were proved, in particular the uniform boundedness principles. The wish to prove the results of this paper was the main motivation for us to work on [13] and [14].

Applications. Let X be a compact C^Q manifold and let $t \mapsto g_t$ be a C^Q-curve of C^Q Riemannian metrics on X. Then we get the corresponding C^Q curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(X)$. By theorem (B) the eigenvalues and eigenvectors can be arranged C^Q. Question: Are the eigenfunctions then also C^Q?

Let Ω be a bounded region in $\mathbb{R}^n$ with C^Q boundary, and let $H(t) = -\Delta + V(t)$ be a C^Q-curve of Schrödinger operators with varying C^Q potential and Dirichlet boundary conditions. Then the eigenvalues and eigenvectors can be arranged C^Q. Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(\Omega \times \mathbb{R})$?

Example. This is an elaboration of [1, 7.4] and [12, Example]. Let $S(2)$ be the vector space of all symmetric real (2×2)-matrices. We use the C^L-curve lemma [13, 3.6] or [14, 2.5]: *There exists a converging sequence of reals t_n with the following property: Let $A_n, B_n \in S(2)$ be any sequences which converge fast to 0, i.e., for each $k \in \mathbb{N}$ the sequences $n^k A_n$ and $n^k B_n$ are bounded in $S(2)$. Then there exists a curve $A \in C^L(\mathbb{R}, S(2))$ such that $A(t_n + s) = A_n + sB_n$ for $|s| \le \frac{1}{n^2}$, for all n.*

We use it for

$$A_n := \frac{1}{2^{n^2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad B_n := \frac{1}{2^{n^2}\, s_n}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \text{where } s_n := 2^{n-n^2} \le \frac{1}{n^2}.$$

The eigenvalues of $A_n + tB_n$ and their derivatives are

$$\lambda_n(t) = \pm\frac{1}{2^{n^2}}\sqrt{1+(\tfrac{t}{s_n})^2}, \quad \lambda_n'(t) = \pm\frac{2^{n^2-2n}t}{\sqrt{1+(\tfrac{t}{s_n})^2}}.$$

Then

$$\begin{aligned}\frac{\lambda'(t_n+s_n)-\lambda'(t_n)}{s_n^\alpha} &= \frac{\lambda_n'(s_n)-\lambda_n'(0)}{s_n^\alpha} = \pm\frac{2^{n^2-2n}s_n}{s_n^\alpha\sqrt{2}} \\ &= \pm\frac{2^{n(\alpha(n-1)-1)}}{\sqrt{2}} \to \infty \text{ for } \alpha > 0.\end{aligned}$$

So condition (in (C), (D), (I), (J), and (K)) that no two unequal continuously parameterized eigenvalues meet of infinite order cannot be dropped. By [1, 2.1], we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues λ are functions which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [19, §2]). Namely, we have

$$A(t_n) = A_n = \frac{1}{2^{n^2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad A(t_n+s_n) = A_n + s_n\, B_n = \frac{1}{2^{n^2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Resolvent Lemma. *Let C^M be any of C^ω, C^Q, C^L, C^∞, or $C^{0,\alpha}$, and let $A(t)$ be normal. If A is C^M then the resolvent $(t,z)\mapsto (A(t)-z)^{-1}\in L(H,H)$ is C^M on its natural domain, the global resolvent set*

$$\{(t,z)\in T\times\mathbb{C} : (A(t)-z) : V\to H \textit{ is invertible}\}$$

which is open (and even connected).

Proof. By definition the function $t\mapsto\langle A(t)v,u\rangle$ is of class C^M for each $v\in V$ and $u\in H$. We may conclude that the mapping $t\mapsto A(t)v$ is of class C^M into H as follows: For $C^M = C^\infty$ we use [11, 2.14.4]. For $C^M = C^\omega$ we use in addition [11, 10.3]. For $C^M = C^Q$ or $C^M = C^L$ we use [14, 2.1] and/or [13, 3.3] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ we use [11, 2.3], [9, 2.6.2], or [8, 4.1.14] because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

For each t consider the norm $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$ on V. Since $A(t)$ is closed, $(V, \|\ \ \|_t)$ is again a Hilbert space with inner product $\langle u,v\rangle_t := \langle u,v\rangle + \langle A(t)u, A(t)v\rangle$.

(1) *Claim (see* [1, in the proof of 7.8]*,* [11, in the proof of 50.16]*, or* [12, Claim 1]*). All these norms $\|\ \ \|_t$ on V are equivalent, locally uniformly in t. We then equip V with one of the equivalent Hilbert norms, say $\|\ \ \|_0$.*

We reduce this to $C^{0,\alpha}$. Namely, note first that $A(t) : (V,\|\ \ \|_s)\to H$ is bounded since the graph of $A(t)$ is closed in $H\times H$, contained in $V\times H$ and thus also closed in $(V,\|\ \ \|_s)\times H$. For fixed $u,v\in V$, the function $t\mapsto\langle u,v\rangle_t = \langle u,v\rangle + \langle A(t)u,A(t)v\rangle$ is $C^{0,\alpha}$ since $t\mapsto A(t)u$ is it. By the multilinear uniform boundedness principle ([11, 5.18] or [9, 3.7.4]) the mapping $t\mapsto\langle\ \ ,\ \ \rangle_t$ is $C^{0,\alpha}$ into the space of bounded sesquilinear forms on $(V,\|\ \ \|_s)$ for each fixed s. Thus the inverse image of $\langle\ \ ,\ \ \rangle_s + \frac12(\text{unit ball})$ in $L(\overline{(V,\|\ \ \|_s)}\oplus(V,\|\ \ \|_s);\mathbb{C})$ is a c^∞-open neighborhood U of s in T. Thus $\sqrt{1/2}\|u\|_s \le \|u\|_t \le \sqrt{3/2}\|u\|_s$ for all $t\in U$, i.e. all Hilbert norms $\|\ \ \|_t$ are locally uniformly equivalent, and claim (1) follows.

By the linear uniform boundedness theorem we see that $t\mapsto A(t)$ is in $C^M(T, L(V,H))$ as follows (here it suffices to use a set of linear functionals which together recognize bounded sets instead of the whole dual): For $C^M = C^\infty$ we use

[11, 1.7 and 2.14.3]. For $C^M = C^\omega$ we use in addition [11, 9.4]. For $C^M = C^Q$ or $C^M = C^L$ we use [14, 2.2 and 2.3] and/or [13, 3.5] where we replace $\mathbb{R}$ by $\mathbb{R}^n$. For $C^M = C^{0,\alpha}$ see above.

If for some $(t,z) \in T \times \mathbb{C}$ the bounded operator $A(t) - z : V \to H$ is invertible, then this is true locally with respect to the c^∞-topology on the product which is the product topology by [11, 4.16], and $(t,z) \mapsto (A(t)-z)^{-1} : H \to V$ is C^M, by the chain rule, since inversion is real analytic on the Banach space $L(V,H)$. □

Note that $(A(t)-z)^{-1} : H \to H$ is a compact operator for some (equivalently any) (t,z) if and only if the inclusion $i : V \to H$ is compact, since $i = (A(t)-z)^{-1} \circ (A(t)-z) : V \to H \to H$.

Polynomial proposition. *Let P be a curve of polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + \cdots + (-1)^n a_n(t), \quad t \in \mathbb{R}.$$

(a) *If P is hyperbolic (all roots real) and if the coefficient functions a_i are all C^Q then there exist C^Q functions λ_i which parameterize all roots.*
(b) *If P is hyperbolic (all roots real), if the coefficient functions a_i are C^L and no two of the different roots meet of infinite order, then there exist C^L functions λ_i which parameterize all roots.*
(c) *If the coefficient functions a_i are C^Q, then for each t_0 there exists $N \in \mathbb{N}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^Q in s for s near 0.*
(d) *If the coefficient functions a_i are C^L and no two of the different roots meet of infinite order, then for each t_0 there exists $N \in \mathbb{N}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^L in s for s near 0.*

All C^Q or C^L solutions differ by permutations.

The proof of parts (a) and (b) is exactly as in [1] where the corresponding results were proven for C^∞ instead of C^L, and for C^ω instead of C^Q. For this we need only the following properties of C^Q and C^L:

- They allow for the implicit function theorem (for [1, 3.3]).
- They contain C^ω and are closed under composition (for [1, 3.4]).
- They are derivation closed (for [1, 3.7]).

Part (a) is also in [6, 7.6] which follows [1]. It also follows from the multidimensional version [17, 6.10] since blow-ups in dimension 1 are trivial. The proofs of parts (c) and (d) are exactly as in [18, 3.2] where the corresponding result was proven for C^ω instead of C^Q, and for C^∞ instead of C^L, if none of the different roots meet of infinite order. For these we need the properties of C^Q and C^L listed above.

Matrix proposition. *Let $A(t)$ for $t \in T$ be a family of $(N \times N)$-matrices.*

(e) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^Q-curve of Hermitian matrices, then the eigenvalues and the eigenvectors can be chosen C^Q.*
(f) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of Hermitian matrices such that no two eigenvalues meet of infinite order, then the eigenvalues and the eigenvectors can be chosen C^L.*
(g) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L-curve of normal matrices such that no two eigenvalues meet of infinite order, then for each t_0 there exists $N_1 \in \mathbb{N}$ such that the eigenvalues and eigenvectors of $s \mapsto A(t_0 \pm s^{N_1})$ can be parameterized C^L in s for s near 0.*
(h) *Let $T \subseteq \mathbb{R}^n$ be open and let $T \ni t \mapsto A(t)$ be a C^ω or C^Q-mapping of normal matrices. Let $K \subseteq T$ be compact. Then there exist a neighborhood W of K, and a finite covering $\{\pi_k : U_k \to W\}$ of W, where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that*

the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s. Consequently, the eigenvalues and eigenvectors of $A(t)$ are locally special functions of bounded variation (SBV). If A is a family of Hermitian matrices, then we do not need power substitutions.

The proof of the matrix proposition in case (e) and (f) is exactly as in [1, 7.6], using the polynomial proposition and properties of C^Q and C^L. Item (g) is exactly as in [18, 6.2], using the polynomial proposition and properties of C^L. Item (h) is proved in [17, 9.1 and 9.6], see also [16].

Proof of the theorem. We have to prove parts (B), (C), (H), (I), (L), (M), and (O). So let C^M be any of C^ω, C^Q, C^L, or $C^{0,\alpha}$, and let $A(t)$ be normal. Let z be an eigenvalue of $A(t_0)$ of multiplicity N. We choose a simple closed C^1 curve γ in the resolvent set of $A(t_0)$ for fixed t_0 enclosing only z among all eigenvalues of $A(t_0)$. Since the global resolvent set is open, see the resolvent lemma, no eigenvalue of $A(t)$ lies on γ, for t near t_0. By the resolvent lemma, $A : T \to L((V, \|\quad\|_0), H)$ is C^M, thus also

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t) - z)^{-1}\, dz =: P(t, \gamma) = P(t)$$

is a C^M mapping. Each $P(t)$ is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of $A(t)$ in the interior of γ, with finite rank. Thus the rank must be constant: It is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in $L(H, H)$ of $P(t)$ to the subset of operators of rank $\le N = \operatorname{rank}(P(t_0))$ is continuous in t and is either 0 or 1.

So for t in a neighborhood U of t_0 there are equally many eigenvalues in the interior of γ, and we may call them $\lambda_i(t)$ for $1 \le i \le N$ (repeated with multiplicity).

Now we consider the family of N-dimensional complex vector spaces $t \mapsto P(t)(H) \subseteq H$, for $t \in U$. They form a C^M Hermitian vector subbundle over U of $U \times H \to U$: For given t, choose $v_1, \dots v_N \in H$ such that the $P(t)v_i$ are linearly independent and thus span $P(t)H$. This remains true locally in t. Now we use the Gram Schmidt orthonormalization procedure (which is C^ω) for the $P(t)v_i$ to obtain a local orthonormal C^M frame of the bundle.

Now $A(t)$ maps $P(t)H$ to itself; in a C^M local frame it is given by a normal $(N \times N)$-matrix parameterized C^M by $t \in U$.

Now all local assertions of the theorem follow:

(B) Use the matrix proposition, part (e).
(C) Use the matrix proposition, part (f).
(H) Use the matrix proposition, part (h), and note that in dimension 1 blowups are trivial.
(I) Use the matrix proposition, part (g).
(L,M) Use the matrix proposition, part (h), for $\mathbb{R}^n$.
(O) We use the following
Result ([5], [4, VII.4.1]) *Let A, B be normal $(N \times N)$-matrices and let $\lambda_i(A)$ and $\lambda_i(B)$ for $i = 1, \dots, N$ denote the respective eigenvalues. Then*

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \le C\|A - B\|$$

for a universal constant C with $1 < C < 3$. Here $\|\quad\|$ is the operator norm.

Finally, it remains to extend the local choices to global ones for the cases (B) and (C) only. There $t \mapsto A(t)$ is C^Q or C^L, respectively, which imply both C^∞, and no two different eigenvalues meet of infinite order. So we may apply [1, 7.8] (in fact we need only the end of the proof) to conclude that the eigenvalues can be chosen C^∞ on $T = \mathbb{R}$, uniquely up to a global permutation. By the local result

above they are then C^Q or C^L. The same proof then gives us, for each eigenvalue $\lambda_i : T \to \mathbb{R}$ with generic multiplicity N, a unique N-dimensional smooth vector subbundle of $\mathbb{R} \times H$ whose fiber over t consists of eigenvectors for the eigenvalue $\lambda_i(t)$. In fact this vector bundle is C^Q or C^L by the local result above, namely the matrix proposition, part (e) or (f), respectively. □

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